

Kernels of Wiener-Hopf plus Hankel operators with matching generating functions

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Dedicated to Roland Duduchava on the occasion of his seventieth birthday

Abstract

Considered are Wiener–Hopf plus Hankel operators $W(a)+H(b) : L^p(\mathbb{R}^+) \rightarrow L^p(\mathbb{R}^+)$ with generating functions a and b from a subalgebra of $L^\infty(\mathbb{R})$ containing almost periodic functions and Fourier images of $L^1(\mathbb{R})$ -functions. If the generating functions a and b satisfy the matching condition

$$a(t)a(-t) = b(t)b(-t), \quad t \in \mathbb{R},$$

an explicit description for the kernels and cokernels of the operators mentioned is given.

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1 Introduction

Let \mathbb{R}^- and \mathbb{R}^+ be, respectively, the subsets of all negative and all positive real numbers, and let χ_E refer to the characteristic function of the subset E of the set of real numbers \mathbb{R} , i.e.

$$\chi_E(t) := \begin{cases} 1 & \text{if } t \in E, \\ 0 & \text{if } t \in \mathbb{R} \setminus E. \end{cases}$$

By $L^p(\mathbb{R}^+) := \chi_{\mathbb{R}^+} L^p(\mathbb{R})$ and $L^p(\mathbb{R}^-) := \chi_{\mathbb{R}^-} L^p(\mathbb{R})$ we denote the subspaces of $L^p(\mathbb{R})$, $1 \leq p \leq \infty$ which contain all functions vanishing on \mathbb{R}^- and \mathbb{R}^+ , correspondingly.

Consider the set G of functions defined on the real line \mathbb{R} and having the form

$$a(t) = \sum_{j=-\infty}^{\infty} a_j e^{i\delta_j t} + \int_{-\infty}^{\infty} k(s) e^{its} ds, \quad -\infty < t < \infty, \quad (1)$$

where δ_j are pairwise distinct real numbers and

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty, \quad \int_{-\infty}^{\infty} |k(s)| ds < \infty.$$

Each element a of G generates three operators $W^0(a) : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ and $W(a), H(a) : L^p(\mathbb{R}^+) \rightarrow L^p(\mathbb{R}^+)$,

$$\begin{aligned} (W^0(a)f)(t) &:= \sum_{j=-\infty}^{\infty} a_j f(t - \delta_j) + \int_{-\infty}^{\infty} k(t-s)f(s) ds, \\ W(a) &:= PW^0(a), \\ H(a) &:= PW^0(a)QJ, \end{aligned}$$

where $P : f \rightarrow \chi_{\mathbb{R}^+} f$ and $Q := I - P$ are the projections on the subspaces $L^p(\mathbb{R}^+)$ and $L^p(\mathbb{R}^-)$, correspondingly, and the operator $J : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ is defined by $J\varphi := \tilde{\varphi}$ with $\tilde{\varphi}(t) := \varphi(-t)$. Note that $W^0(a), W(a)$ and $H(a)$ are bounded linear operators on the corresponding spaces. The function a is called the generating function or the symbol for each of the operators $W^0(a), W(a)$ and $H(a)$. Wiener-Hopf and Hankel operators are closely connected. Thus for any $a, b \in G$, one has

$$\begin{aligned} W(ab) &= W(a)W(b) + H(a)H(\tilde{b}), \\ H(ab) &= W(a)H(b) + H(a)W(\tilde{b}). \end{aligned} \quad (2)$$

The Fredholm theory for the operators $W^0(a)$, $a \in G$ is relatively simple. An operator $W^0(a)$ is semi-Fredholm if and only if a is invertible in G . The study of the operators $W(a)$ is more involved. Nevertheless, for various classes of generating functions a , Wiener-Hopf operators $W(a)$ are well studied (see, for example, [2, 3, 7, 12, 13, 14, 15]). In particular, Fredholm properties of such operators are known and a description of the kernel is available. On the other hand, Wiener-Hopf plus Hankel operators, i.e. the operators of the form

$$B = B(a, b) = W(a) + H(b)$$

remains less studied. Fredholm properties of such operators can be derived by reducing the initial operator to a Wiener-Hopf operator with a matrix symbol, and there is a number of works where this idea is successfully implemented [1, 4, 5, 6]. However, these works mainly deal with generating functions a and b satisfying the condition $a = b$ and consider the operators acting in an L^2 -space. If $a \neq b$, then a rarely verifiable assumption about special matrix factorization is used. A different approach to the study of the operators of the form $I + H(b)$ has been employed in [16, 17], where the essential spectrum and the index of such operators have been found. On the other hand, no information is available about the kernel elements of the operators $W(a) + H(b)$ even in the above mentioned cases $a = 1$ and $a = b$. The goal of this work is to present an efficient description of the space $\ker B(a, b)$ when the generating functions a and b belong to the Banach algebra G and satisfy a specific algebraic relation. Point out that our approach does not involve factorization of any matrix function but only the one of scalar functions.

Let $a, b \in L^\infty(\mathbb{R})$. We say that the duo (a, b) is a matching pair if

$$a\tilde{a} = b\tilde{b}, \quad (3)$$

where $\tilde{a} := a(-t)$. The relation (3) is called matching condition. In the following we always assume that a , and therefore b , is invertible in G . For each matching pair (a, b) , consider the pair (c, d) with

$$c := \tilde{b}\tilde{a}^{-1}, \quad d := b\tilde{a}^{-1}.$$

It is easily seen that (c, d) is also a matching pair. This pair is called the subordinated pair for (a, b) or just the subordinated pair. The elements c and d of the subordinated pair possess a specific property, namely

$$c\tilde{c} = 1, \quad d\tilde{d} = 1.$$

Throughout this paper any function $g \in G$ satisfying the condition

$$g\tilde{g} = 1,$$

is called matching function. Note that the matching functions c and d can be also expressed as

$$c = ab^{-1}, \quad d = a\tilde{b}^{-1}.$$

Further, if (c, d) is the subordinated pair for (a, b) , then (\bar{d}, \bar{c}) is the subordinated pair for the matching pair (\bar{a}, \bar{b}) . Moreover, if $p \in [1, \infty)$, then \bar{a} and \bar{b} are generating functions for the operator adjoint to the Wiener-Hopf plus Hankel operator $W(a) + H(b) : L^p(\mathbb{R}^+) \rightarrow L^p(\mathbb{R}^+)$, i.e.

$$(W(a) + H(b))^* = W(\bar{a}) + H(\bar{b}). \quad (4)$$

The Wiener-Hopf operators with matching generating symbols possess a number of remarkable properties. In particular, the kernels of such operators can be structured in a special way and this structurization can be used in the description of the kernels of Wiener-Hopf plus Hankel operators. More precisely, let g be a matching function and let $\mathbf{P}(g)$ be the operator defined on the kernel $\ker W(g)$ by

$$\mathbf{P}(g) := JQW^0(g)P \big|_{\ker W(g)}. \quad (5)$$

One can easily check that $\mathbf{P}(g)$ maps $\ker W(g)$ into $\ker W(g)$ and $\mathbf{P}^2(g) = I$ (see [8] for more details). Therefore, the operators

$$\mathbf{P}^-(g) := (1/2)(I - \mathbf{P}(g)), \quad \mathbf{P}^+(g) := (1/2)(I + \mathbf{P}(g)), \quad (6)$$

considered on the space $\ker W(g)$, are complementary projections generating a decomposition of $\ker W(g)$, i.e.

$$\ker W(g) = \text{im } \mathbf{P}^-(g) \dot{+} \text{im } \mathbf{P}^+(g).$$

Consider now the Wiener-Hopf plus Hankel operators $W(a) + H(b)$, generating functions of which constitute a matching pair. In this case the elements of the subordinated pair (c, d) are matching functions. Assume that the operator $W(c)$ is right-invertible and let W_r^{-1} be a right inverse for $W(c)$. By φ_{\pm} we denote the operators defined on the kernel of the operator $W(d)$ by

$$2\varphi_{\pm}(s) := W_r^{-1}(c)W(\tilde{a}^{-1})s \mp JQW^0(c)PW_r^{-1}(c)W(\tilde{a}^{-1})s \pm JQW^0(\tilde{a}^{-1})s, \quad (7)$$

where $\tilde{a}^{-1} = a^{-1}(-t)$. It was shown in [8] that for any $s \in \ker W(d)$ one has $\varphi_{\pm}(s) \in \ker(W(a) \pm H(b))$, and the operators φ_+ and φ_- are injections on the spaces $\text{im } \mathbf{P}^+(d)$ and $\text{im } \mathbf{P}^-(d)$, respectively. Moreover, the following result is true.

Proposition 1.1 (see [8, Proposition 2.3]) *Let (c, d) be the subordinated pair for a matching pair $(a, b) \in G \times G$. If the operator $W(c)$ is right-invertible, then*

$$\begin{aligned} \ker(W(a) + H(b)) &= \varphi_+(\text{im } \mathbf{P}^+(d)) \dot{+} \text{im } \mathbf{P}^-(c), \\ \ker(W(a) - H(b)) &= \varphi_-(\text{im } \mathbf{P}^-(d)) \dot{+} \text{im } \mathbf{P}^+(c). \end{aligned} \quad (8)$$

Thus to describe the kernels of the Wiener-Hopf plus/minus Hankel operators, one needs to find an efficient description of the images of the projections $\mathbf{P}^{\pm}(c)$ and $\mathbf{P}^{\pm}(d)$. Notice that the above statements do not depend on p .

This paper is organized as follows. In Section 2 we present a decomposition of the kernel of $W(g)$ with a generating matching function g . These results are used in Section 3 in order to derive an efficient description of the kernels $\ker(W(a) \pm H(b))$, $p \in [1, \infty]$ and the cokernels $\text{coker}(W(a) \pm H(b))$, $p \in [1, \infty)$. Similar results for Toeplitz plus Hankel operators have been obtained in [9, 10], and generalized Toeplitz plus Hankel operators are considered in [11]. However, all the relevant operators in [9, 10, 11] are Fredholm. On the other hand, the really new feature of the present study is the consideration of situations where the operators $W(c)$ and $W(d)$ can have infinite-dimensional kernels or co-kernels.

2 Kernels of Wiener-Hopf operators with a matching generating function.

Our aim now is to describe the subspaces $\text{im } \mathbf{P}^\pm(g) \subset \ker W(g)$. For, let us recall certain results of Fredholm theory for Wiener-Hopf operators with generating functions from the Banach algebra G . As we know, any element $a \in G$ can be represented in the form $a = b + k$, where b belongs to the algebra AP_w of all almost periodic functions with absolutely convergent Fourier series and k is in the algebra \mathcal{L}_0 of all Fourier transforms of functions from $L^1(\mathbb{R})$. If $a = b + k$, $b \in AP_w$, $k \in \mathcal{L}_0$ is an invertible element of G , then b is invertible in AP_w and one can define the numbers $\nu = \nu(a)$ and $n = n(a)$ by

$$\nu(a) := \lim_{l \rightarrow \infty} \frac{1}{2l} [\arg b(t)]_{-l}^l, \quad n(a) := \frac{1}{2\pi} [\arg(1 + b^{-1}(t)k(t))]_{t=-\infty}^{\infty}.$$

Recall that $a \in G$ is invertible in G if and only if $\inf_{t \in \mathbb{R}} |a(t)| > 0$ and \mathcal{L}_0 forms a closed two-sided ideal in G .

Theorem 2.1 (Gohberg/Feldman [15]) *Let $1 \leq p \leq \infty$ and $g \in G$. The operator $W(g)$ is one-sided invertible in the space $L^p(\mathbb{R}^+)$ if and only if g is invertible in G . Further, if $g \in G$ is invertible in G , then the following assertions are true:*

- (i) *If $\nu(g) < 0$, then the operator $W(g)$ is invertible from the right and $\dim \ker W(g) = \infty$.*
- (ii) *If $\nu(g) = 0$ and $n(g) \geq 0$ ($\nu(g) = 0$ and $n(g) \leq 0$), then the operator $W(g)$ is invertible from the left (from the right) and*

$$\dim \text{coker } W(g) = n(g) \quad (\dim \ker W(g) = -n(g)).$$

- (iii) *If $\nu(g) > 0$, then the operator $W(g)$ is invertible from the left and $\dim \text{coker } W(g) = \infty$.*

- (iv) *If $g \in G$ is not invertible in G , then $W(g)$ is not a semi-Fredholm operator.*

The proof of this theorem is based on the fact that every invertible function $a \in G$ admits a factorization of the form

$$g(t) = g_-(t) e^{i\nu t} \left(\frac{t-i}{t+i} \right)^n g_+(t), \quad -\infty < t < \infty, \quad (9)$$

where $g_+^{\pm 1} \in G^+$, $g_-^{\pm 1} \in G^-$, $\nu = \nu(g)$ and $n = n(g)$. Recall that $G^+(G^-)$ is defined as follows: $G^+(G^-)$ consists of all functions (1) such that all indices δ_j are non-negative (non-positive) and the function k vanishes on the negative (positive)

semi-axis. It is clear that functions from G^+ and G^- admit holomorphic extensions to the upper and to the lower half-plane, correspondingly, and the intersection of the algebras G^+ and G^- consists of constant functions only. Note that under the condition $g_-(0) = 1$, the factorization (9) is unique. Moreover, for $a \in G^-$, $b \in G$ and $c \in G^+$, the first identity from (2) leads to the relation

$$W(abc) = W(a)W(b)W(c).$$

Combined with the factorization (9), this relation leads to the following representation of the operator $W(g)$,

$$W(g) = W(g_-)W\left(e^{i\nu t}\left(\frac{t-i}{t+i}\right)^n\right)W(g_+).$$

Therefore, theory of the Wiener-Hopf operators $W(g)$ with invertible symbol g is based on the study of the middle factor of this factorization (see [15, Chapter VII]). Thus the operator $W(a)$ has a kernel containing non-zero elements in the two cases—viz. if $\nu < 0$, then $\dim \ker W(g) = \infty$, or if $\nu = 0$ and $n < 0$, then $\dim \ker W(g) = |n|$. In what follows we consider all possible situations separately. Let us note that $\ker W(a)$ do not depend on p .

Assume that g is a matching function. Then, as was pointed out in [8], the factorization (9) comes down to the following one

$$g(t) = \sigma(g) \tilde{g}_+^{-1}(t) e^{i\nu t} \left(\frac{t-i}{t+i}\right)^n g_+(t) \quad (10)$$

where $\sigma(g) = (-1)^n g(0)$, $\tilde{g}_+^{\pm 1}(t) \in G^-$ and $g_-(t) = \sigma(g) \tilde{g}_+^{-1}(t)$. In passing note that $\sigma(g) = \pm 1$.

Our goal now is to describe the projections $\mathbf{P}^\pm(g)$ from (6). Let us start with the case where the parameters ν and n in the factorization (10) satisfy the relations $\nu = 0$, $n < 0$. It is known [15] that in this case

$$\ker W(g) = \left\{ W(g_+^{-1}) \left(\sum_{j=0}^{|n|-1} c_j t^j e^{-t} \right) : c_j \in \mathbb{C} \right\}.$$

Thus the functions $W(g_+^{-1}) t^j e^{-t}$, $j = 0, 1, \dots, |n| - 1$ form a basis in $\ker W(g)$. However, the space $\ker W(g)$ has another basis, namely,

$$\{W(g_+^{-1})\psi_j(t) : j = 0, 1, \dots, |n| - 1\}, \quad (11)$$

where

$$\psi_j(t) := \begin{cases} \sqrt{2}e^{-t}\Lambda_j(2t), & \text{if } t > 0, \\ 0, & \text{if } t < 0, \end{cases}, \quad j = 0, 1, \dots,$$

and Λ_j are the normalized Laguerre polynomials. Moreover, for $j = -1, -2, \dots$, one can define the functions ψ_j by

$$\psi_j(t) := -\psi_{-j-1}(-t), \quad j = -1, -2, \dots, \quad (12)$$

The functions ψ_j , $j \in \mathbb{Z}$ can be also expressed in the form

$$\begin{aligned} \psi_j(t) &= (U^j \psi_0)(t), \quad j = \pm 1, \pm 2, \dots, \\ \psi_0(t) &= \begin{cases} \sqrt{2}e^{-t}, & \text{if } t > 0 \\ 0, & \text{if } t < 0, \end{cases} \end{aligned} \quad (13)$$

where $U := W^0((\lambda - i)/(\lambda + i))$. Note that the operators U^j , $j \in \mathbb{Z}$ are unitary operators on $L^2(\mathbb{R})$. Thus, the functions ψ_j , $j \in \mathbb{Z}$ form an orthonormal basis on this space. Indeed, it is shown in [15, Chapter 3, §3.2] that for $j > 0$, one has

$$(U^j \psi_0)(t) = \psi_j(t),$$

and applying (12) one gets the result. Note that the relation (12) can be obtained by using the Fourier transform. Indeed, let us recall the formula

$$(\mathcal{F}\psi_n)(\lambda) = \int_0^\infty \psi_n(t) e^{i\lambda t} dt = \int_0^\infty U^n \psi_0(t) e^{i\lambda t} dt = \left(\frac{\lambda - i}{\lambda + i} \right)^n \frac{i\sqrt{2}}{\lambda + i}, \quad n \in \mathbb{Z}_+,$$

where \mathcal{F} is the Fourier transform [15] and \mathbb{Z}_+ refers to the set of all non-negative integers. Consider the operator $J : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$ defined by $(Jf)(t) = f(-t)$. If $n \in \mathbb{N}$, then one has

$$\begin{aligned} \mathcal{F}(-J\psi_{n-1})(\lambda) &= -J\mathcal{F}(\psi_{n-1})(\lambda) \\ &= -J \left(\left(\frac{\lambda - i}{\lambda + i} \right)^{n-1} \frac{i\sqrt{2}}{\lambda + i} \right) = - \left(\left(\frac{-\lambda - i}{-\lambda + i} \right)^{n-1} \frac{i\sqrt{2}}{-\lambda + i} \right) \\ &= \left(\frac{\lambda + i}{\lambda - i} \right)^{n-1} \frac{i\sqrt{2}}{\lambda - i}. \end{aligned} \quad (14)$$

On the other hand, if $n \leq -1$, then

$$\begin{aligned} (\mathcal{F}\psi_n)(\lambda) &= \int_0^\infty \psi_n(t) e^{i\lambda t} dt = \int_0^\infty U^n \psi_0(t) e^{i\lambda t} dt \\ &= \left(\frac{\lambda + i}{\lambda - i} \right)^{|n|} \frac{i\sqrt{2}}{\lambda + i} = \left(\frac{\lambda + i}{\lambda - i} \right)^{|n|-1} \frac{i\sqrt{2}}{\lambda - i}. \end{aligned} \quad (15)$$

Comparing (14) and (15), one obtains that

$$\mathcal{F}(\psi_n(t)) = \mathcal{F}(-\psi_{|n|-1}(-t))$$

and one has to use the injectivity of the Fourier transform to complete the proof.

Let g be a matching function. In order to describe the corresponding projections $\mathbf{P}^\pm(g)$ of (5)-(6), we will study how the operator $\mathbf{P}(g)$ interacts with the basis elements (11). Thus

$$\begin{aligned}\mathbf{P}(g)W(g_+^{-1})\psi_j(t) &= JQW^0(g)PW(g_+^{-1})\psi_j(t) \\ &= JQW^0\left(\sigma(g)\tilde{g}_+^{-1}\left(\frac{t-i}{t+i}\right)^{-|n|}g_+\right)W(g_+^{-1})\psi_j \\ &= \sigma(g)JQW^0(\tilde{g}_+^{-1})W^0\left(\left(\frac{t-i}{t+i}\right)^{-|n|}\right)\psi_j.\end{aligned}$$

Considering the elements $W^0\left(\left((t-i)/(t+i)\right)^{-|n|}\right)\psi_j$, $j = 0, 1, \dots, |n|-1$ and using relations (12) and (13), we get

$$\begin{aligned}W^0\left(\left(\frac{t-i}{t+i}\right)^{-|n|}\right)\psi_j &= W^0\left(\left(\frac{t-i}{t+i}\right)^{-|n|}\right)W^0\left(\left(\frac{t-i}{t+i}\right)^j\right)\psi_0 \\ &= W^0\left(\left(\frac{t-i}{t+i}\right)^{-|n|+j}\right)\psi_0 = \psi_{-|n|+j} = -J\psi_{|n|-j-1}.\end{aligned}$$

Hence,

$$\sigma(g)JQW^0(\tilde{g}_+^{-1})W^0\left(\left(\frac{t-i}{t+i}\right)^{-|n|}\right)\psi_j = -\sigma(g)PW^0(g_+^{-1})\psi_{|n|-j-1}.$$

Now one can proceed similarly to [10, Section 5] and obtain the following result.

Theorem 2.2 *Let $g \in G$ be a matching function such that the operator $W(g) : L_p(\mathbb{R}_+) \rightarrow L_p(\mathbb{R}_+)$ is Fredholm and $n := \text{ind } W(g) > 0$. If*

$$g(t) = g_-(t)\left(\frac{t-i}{t+i}\right)^{-n}g_+(t) = \sigma(g)\tilde{g}_+^{-1}(t)\left(\frac{t-i}{t+i}\right)^{-n}g_+(t), \quad g_-(0) = 1,$$

is the related Wiener-Hopf factorization of the function g , then the following systems $\mathfrak{B}_\pm(g)$ of functions $W(g_+^{-1})\psi_j$ form bases in the spaces $\text{im } \mathbf{P}^\pm(g)$:

(i) *If $n = 2m$, $m \in \mathbb{N}$, then*

$$\mathfrak{B}_\pm(g) = \{W(g_+^{-1})(\psi_{m-k-1} \mp \sigma(g)\psi_{m+k}) : k = 0, 1, \dots, m-1\},$$

and

$$\dim \text{im } \mathbf{P}^\pm(g) = m.$$

(ii) If $n = 2m + 1$, $m \in \mathbb{Z}_+$, then

$$\mathfrak{B}_\pm(g) = \{W(g_+^{-1})(\psi_{m+k} \mp \sigma(g)\psi_{m-k}) : k = 0, 1, \dots, m\},$$

and

$$\dim \operatorname{im} \mathbf{P}^\pm(g) = m + \frac{1 \mp \sigma(g)}{2}.$$

Remark 2.1 It is worth mentioning that the zero element belongs to the one of the sets $\{W(g_+^{-1})(\psi_{m+k} - \sigma(g)\psi_{m-k}) : k = 0, 1, \dots, m\}$ or $\{W(g_+^{-1})(\psi_{m+k} + \sigma(g)\psi_{m-k}) : k = 0, 1, \dots, m\}$ only. Namely, for $k = 0$ one of the terms $\psi_m(1 \pm \sigma(g))$ is equal to zero.

Consider now the case $\nu < 0$ and $n = 0$. Then

$$\ker W(g) = \{W(g_+^{-1})f : f \in L^p(\mathbb{R}^+) \text{ and } f(t) = 0 \text{ for } t > |\nu|\},$$

(see [15, Chapter VII, §2.4]).

Theorem 2.3 Let $g \in G$ be a matching function such that the function g possesses the Wiener-Hopf factorization

$$g(t) = g_-(t)e^{i\nu t}g_+(t) = \sigma(g)\tilde{g}_+^{-1}(t)e^{i\nu t}g_+(t), \quad \nu < 0 \text{ and } g_-(0) = 1,$$

and let $h \in \ker W(g)$, that is $h = W(g_+^{-1})f$ with an $f \in L^p(\mathbb{R}^+)$ such that $f(t) = 0$ for $t > |\nu|$. Then

$$JQW^0(g)Ph = \sigma(g)W(g_+^{-1})\mathcal{R}_{|\nu|}f,$$

where

$$(\mathcal{R}_{|\nu|})(t) = \begin{cases} f(|\nu| - t), & \text{if } 0 < t < |\nu| \\ 0 & \text{if } t > |\nu| \end{cases}, \quad (16)$$

and

$$\mathbf{P}^\pm(g)h = \frac{h \pm \sigma(g)W(g_+^{-1})\mathcal{R}_{|\nu|}f}{2}.$$

The proof of this result runs similarly to the proof of Theorem 2.5 below where a more general factorization of the corresponding matching function g has to be used.

Next we consider the situation $\nu < 0$ and $n < 0$. In this case the function g admits the Wiener-Hopf factorization of the form

$$g = \sigma(g)\tilde{g}_+^{-1}e^{i\nu t}\left(\frac{t-i}{t+i}\right)^n g_+. \quad (17)$$

As is shown in [15, Chapter VII], the kernel of the operator $W(g)$ is the direct sum of the kernels of the operators $W(g_+((t-i)/(t+i))^n)$ and $W(g_+e^{i\nu t})$. Thus

$$\ker W(g) = \ker W\left(g_+\left(\frac{t-i}{t+i}\right)^n\right) \dot{+} \ker W(g_+e^{i\nu t}).$$

Therefore, in order to characterize the projections $\mathbf{P}^\pm(g) : \ker W(g) \rightarrow \ker W(g)$, one can describe their action on the subspaces $\ker W(g_+((t-i)/(t+i))^n)$ and $\ker W(g_+e^{i\nu t})$ separately. To this aim, let us use the following representations of the function g :

$$\begin{aligned} g &= e^{i\nu t} g_1, \quad g_1 := \boldsymbol{\sigma}(g) \widetilde{g}_+^{-1} \left(\frac{t-i}{t+i} \right)^n g_+, \\ g &= \left(\frac{t-i}{t+i} \right)^n g_2, \quad g_2 := \boldsymbol{\sigma}(g) \widetilde{g}_+^{-1} e^{i\nu t} g_+. \end{aligned}$$

Moreover, observe that $JQW^0(g)P = H(\widetilde{g})$.

Theorem 2.4 *Assume that g is a matching function of the form (17).*

(i) *If $h \in \ker W(g_+((t-i)/(t+i))^n)$, then*

$$\mathbf{P}^\pm(g)h = \frac{1}{2} [I \pm (W(e^{i|\nu|t})(\mathbf{P}^+(g_1) - \mathbf{P}^-(g_1)))] h.$$

(ii) *If $h \in \ker W(g_+e^{i\nu t})$, then*

$$\mathbf{P}^\pm(g)h = \frac{1}{2} \left[I \pm \left(W \left(\left(\frac{t-i}{t+i} \right)^{|n|} \right) (\mathbf{P}^+(g_2) - \mathbf{P}^-(g_2)) \right) \right] h.$$

Proof. Let us start with assertion (i). Using (2) we obtain

$$JQW^0(g)P = PW^0(\widetilde{g})QJ = H(\widetilde{g}) = W(\widetilde{e^{i\nu t}}) H(\widetilde{g}_1) + H(\widetilde{e^{i\nu t}}) W(g_1),$$

and the relation $W(g_1)h = 0$ implies that

$$H(\widetilde{g})h = W(e^{i|\nu|t}) H(\widetilde{g}_1)h.$$

Therefore,

$$\begin{aligned} \mathbf{P}^\pm(g)h &= \left[\frac{I \pm H(\widetilde{g})}{2} \right] h = \left[\frac{I \pm W(e^{i|\nu|t}) H(\widetilde{g}_1)}{2} \right] h \\ &= \frac{1}{2} [I \pm (W(e^{i|\nu|t})(\mathbf{P}^+(g_1) - \mathbf{P}^-(g_1)))] h, \end{aligned}$$

so the assertion (i) is proved.

The proof of assertion (ii) is similar to that of (i). It is based on the formula

$$H(\widetilde{g}) = W \left(\left(\frac{t-i}{t+i} \right)^{|n|} \right) H(\widetilde{g}_2) + H \left(\left(\frac{t-i}{t+i} \right)^{|n|} \right) W(g_2),$$

and is left to the reader. ■

Remark 2.2 Recall that the projections $\mathbf{P}^\pm(g_1)$ and $\mathbf{P}^\pm(g_2)$ acting, respectively, on the subspaces $\ker W(g_1)$ and $\ker W(g_2)$ are described by Theorem 2.2 and Theorem 2.3. Besides,

$$\sigma(g) = \sigma(g_1) = \sigma(g_2).$$

Finally, let us consider the case $\nu < 0$ and $n > 0$, i.e. now we assume that the Wiener-Hopf factorization of the matching function g is

$$g = \sigma(g) \tilde{g}_+^{-1} e^{i\nu t} \left(\frac{t-i}{t+i} \right)^n g_+. \quad (18)$$

If this is the case, then according to [15, Chapter VII] the kernel of the operator $W(g)$ consists of the functions h having the form

$$h = W(g_+^{-1}) W \left(\left(\frac{t-i}{t+i} \right)^{-n} \right) \varphi, \quad (19)$$

where $\varphi \in L^p(\mathbb{R}^+)$ is such that

$$\varphi(t) = 0 \text{ for all } t > |\nu| \text{ and } \int_0^\infty \varphi(t) t^j e^{-t} dt = 0, \quad j = 0, 1, \dots, n-1. \quad (20)$$

Theorem 2.5 Let $g \in G$ be a matching function such that the function g possesses the Wiener-Hopf factorization (18). Assume that $h \in \ker W(g)$. Then it can be represented in the form (19)–(20) and

$$JQW^0(g)Ph = \sigma(g) W(g_+^{-1}) \mathcal{R}_{|\nu|} \varphi,$$

where $\mathcal{R}_{|\nu|}$ is defined by (16) and

$$\mathbf{P}^\pm(g)h = \frac{h \pm \sigma(g) W(g_+^{-1}) \mathcal{R}_{|\nu|} \varphi}{2}.$$

Proof. Consider the expression $JQW^0(g)Ph$. One has

$$\begin{aligned} JQW^0(g)Ph &= \sigma(g) JQW^0(\tilde{g}_+^{-1}) W^0(e^{i\nu t}) W^0 \left(\left(\frac{t-i}{t+i} \right)^n \right) W^0(g_+) Ph \\ &= \sigma(g) JQW^0(\tilde{g}_+^{-1}) W^0(e^{i\nu t}) P\varphi = \sigma(g) PW^0(g_+^{-1}) W^0(e^{i|\nu|t}) JP\varphi \\ &= \sigma(g) PW(g_+^{-1}) \mathcal{R}_{|\nu|} \varphi. \end{aligned}$$

Application of the relation

$$\mathbf{P}^\pm(g)h = \frac{h \pm JQW(g)Ph}{2},$$

competes the proof. ■

3 Kernels and cokernels of Wiener-Hopf plus Hankel operators. Specification.

In this section we study the kernels and cokernels of Wiener-Hopf plus Hankel operators in the case where the generating functions $a, b \in G$ satisfy the matching condition (3) and a is invertible in G . Then according to Theorem 2.1, the operators $W(c)$ and $W(d)$ are one-sided invertible in $L^p(\mathbb{R}^+)$, $1 \leq p \leq \infty$. Using results of Section 2, we derive an explicit description for the kernels and cokernels of the operators mentioned. As before, we again have to consider several cases.

3.1 The Case I: $\nu(c) = \nu(d) = 0$.

This case is also used as a model case in order to show how to handle all other situations. If the indexes $\nu(c)$ and $\nu(d)$ are equal to zero, then the operators $W(c)$ and $W(d)$ are Fredholm. Using the relations (2.4) and (2.7) of [8], one obtains that the operators $W(a) \pm H(b)$ are Fredholm. Set $\kappa_1 := \text{ind } W(c)$, $\kappa_2 := \text{ind } W(d)$ and let \mathbb{Z}_- and \mathbb{Z}_+ refer to the set of all negative and non-negative integers, correspondingly.

Theorem 3.1 *Assume that $\nu(c) = \nu(d) = 0$.*

- (i) *If $(\kappa_1, \kappa_2) \in \mathbb{Z}_+ \times \mathbb{N}$, then for all $p \in [1, \infty]$ the operators $W(a) \pm H(b) : L^p \rightarrow L^p$ are invertible from the right and*

$$\begin{aligned} \ker(W(a) + H(b)) &= \text{im } \mathbf{P}^-(c) \dot{+} \varphi_+(\text{im } \mathbf{P}^+(d)), \\ \ker(W(a) - H(b)) &= \text{im } \mathbf{P}^+(c) \dot{+} \varphi_-(\text{im } \mathbf{P}^-(d)), \end{aligned}$$

where the spaces $\text{im } \mathbf{P}^\pm(c)$, $\text{im } \mathbf{P}^\pm(d)$ are described in Theorem 2.2 and the mappings φ_\pm are defined by (7).

- (ii) *If $(\kappa_1, \kappa_2) \in \mathbb{Z}_- \times (\mathbb{Z} \setminus \mathbb{N})$, then for all $p \in [1, \infty]$ the operators $W(a) \pm H(b) : L^p \rightarrow L^p$ are invertible from the left and for all $p \in [1, \infty)$ one has*

$$\begin{aligned} \text{coker}(W(a) + H(b)) &= \text{im } \mathbf{P}^-(\bar{d}) \dot{+} \varphi_+(\text{im } \mathbf{P}^+(\bar{c})), \\ \text{coker}(W(a) - H(b)) &= \text{im } \mathbf{P}^+(\bar{d}) \dot{+} \varphi_-(\text{im } \mathbf{P}^-(\bar{c})), \end{aligned}$$

with $\text{im } \mathbf{P}^\pm(\bar{d}) = \{0\}$ if $\kappa_2 = 0$.

- (iii) *If $(\kappa_1, \kappa_2) \in \mathbb{Z}_+ \times (\mathbb{Z} \setminus \mathbb{N})$, then for all $p \in [1, \infty]$ one has*

$$\begin{aligned} \ker(W(a) + H(b)) &= \text{im } \mathbf{P}^-(c), \\ \ker(W(a) - H(b)) &= \text{im } \mathbf{P}^+(c), \end{aligned}$$

and for all $p \in [1, \infty)$,

$$\begin{aligned}\operatorname{coker}(W(a) + H(b)) &= \operatorname{im} \mathbf{P}^-(\bar{d}), \\ \operatorname{coker}(W(a) - H(b)) &= \operatorname{im} \mathbf{P}^+(\bar{d}).\end{aligned}$$

Proof. Let us note that all results concerning the kernels of the corresponding operators follow immediately from Proposition 1.1 and from Theorem 2.2. As far as the cokernel structure is concerned, one has to take into account the already mentioned relation (4) and the fact that (\bar{d}, \bar{c}) is the subordinated pair for the duo (\bar{a}, \bar{b}) . \blacksquare

It remains to consider the case $(\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{N}$. This situation is more involved. In order to formulate the next result, we need a special representation for the index of the operator $W(c)$. Thus chose $k \in \mathbb{N}$ such that

$$1 \geq 2k + \kappa_1 \geq 0.$$

Such a number k is uniquely defined and

$$2k + \kappa_1 = \begin{cases} 0, & \text{if } \kappa_1 \text{ is even,} \\ 1, & \text{if } \kappa_1 \text{ is odd.} \end{cases}$$

Now the operators $W(a) \pm H(b)$ can be represented in the form

$$W(a) \pm H(b) = \left(W \left(a \left(\frac{t-i}{t+i} \right)^{-k} \right) \pm H \left(b \left(\frac{t-i}{t+i} \right)^k \right) \right) W \left(\left(\frac{t-i}{t+i} \right)^k \right). \quad (21)$$

Observe that $\left(a \left(\frac{t-i}{t+i} \right)^{-k}, b \left(\frac{t-i}{t+i} \right)^k \right)$ is a matching pair with the subordinated pair $\left(c \left(\frac{t-i}{t+i} \right)^{-2k}, d \right)$. Therefore, the operators $W \left(a \left(\frac{t-i}{t+i} \right)^{-k} \right) \pm H \left(b \left(\frac{t-i}{t+i} \right)^k \right)$ are subject to assertion (i) of Theorem 3.1. Thus they are right-invertible, and if κ_1 is even, then

$$\begin{aligned}\ker \left(W \left(a \left(\frac{t-i}{t+i} \right)^{-k} \right) + H \left(b \left(\frac{t-i}{t+i} \right)^k \right) \right) &= \varphi_+(\operatorname{im} \mathbf{P}^+(d)), \\ \ker \left(W \left(a \left(\frac{t-i}{t+i} \right)^{-k} \right) - H \left(b \left(\frac{t-i}{t+i} \right)^k \right) \right) &= \varphi_-(\operatorname{im} \mathbf{P}^-(d)),\end{aligned} \quad (22)$$

and if κ_1 is odd, then

$$\begin{aligned}
& \ker \left(W \left(a \left(\frac{t-i}{t+i} \right)^{-k} \right) + H \left(b \left(\frac{t-i}{t+i} \right)^k \right) \right) \\
& \quad = \frac{1 - \sigma(c)}{2} W(c_+^{-1}) \{ \mathbb{C}\psi_0 \} \dot{+} \varphi_+(\text{im } \mathbf{P}^+(d)), \\
& \ker \left(W \left(a \left(\frac{t-i}{t+i} \right)^{-k} \right) - H \left(b \left(\frac{t-i}{t+i} \right)^k \right) \right) \\
& \quad = \frac{1 + \sigma(c)}{2} W(c_+^{-1}) \{ \mathbb{C}\psi_0 \} \dot{+} \varphi_-(\text{im } \mathbf{P}^-(d)),
\end{aligned} \tag{23}$$

where the function ψ_0 is defined by (13) and the mappings φ_{\pm} depend on the functions $a \left(\frac{t-i}{t+i} \right)^{-k}$ and $b \left(\frac{t-i}{t+i} \right)^k$.

Theorem 3.2 *Let $(\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{N}$ and $p \in [1, \infty)$. Then*

(i) *If κ_1 is odd, then*

$$\begin{aligned}
& \ker(W(a) \pm H(b)) = W \left(\left(\frac{t-i}{t+i} \right)^{-k} \right) \\
& \quad \times \left(\left\{ \frac{1 \mp \sigma(c)}{2} W(c_+^{-1}) \{ \mathbb{C}\psi_0 \} \dot{+} \varphi_{\pm}(\text{im } \mathbf{P}^{\pm}(d)) \right\} \cap \text{im } W \left(\left(\frac{t-i}{t+i} \right)^k \right) \right) \\
& \quad = \left\{ \psi \in \left\{ W \left(\left(\frac{t-i}{t+i} \right)^{-k} \right) u : u \in \left\{ \frac{1 \mp \sigma(c)}{2} W(c_+^{-1}) \{ \mathbb{C}\psi_0 \} \dot{+} \varphi_{\pm}(\text{im } \mathbf{P}^{\pm}(d)) \right\} \right. \right. \\
& \quad \quad \left. \left. \text{and } \int_0^{\infty} u(t) e^{-t} t^j dt = 0 \text{ for all } j = 0, 1, \dots, k-1, \right\} \right\},
\end{aligned}$$

where the mappings φ_{\pm} depend on the functions $a \left(\frac{t-i}{t+i} \right)^{-k}$ and $b \left(\frac{t-i}{t+i} \right)^k$. The last means that the functions a, b and c in the expression (7) have to be, respectively, replaced by $a \left(\frac{t-i}{t+i} \right)^{-k}$, $b \left(\frac{t-i}{t+i} \right)^k$ and $c \left(\frac{t-i}{t+i} \right)^{-2k}$.

(ii) *If κ_1 is even, then*

$$\begin{aligned}
& \ker(W(a) \pm H(b)) = W \left(\left(\frac{t-i}{t+i} \right)^{-k} \right) \left(\{ \varphi_{\pm}(\text{im } \mathbf{P}^{\pm}(d)) \} \cap \text{im } W \left(\left(\frac{t-i}{t+i} \right)^k \right) \right) \\
& \quad = \left\{ \psi \in \left\{ W \left(\left(\frac{t-i}{t+i} \right)^{-k} \right) u : u \in \{ \{ \mathbb{C}\psi_0 \} \dot{+} \varphi_{\pm}(\text{im } \mathbf{P}^{\pm}(d)) \} \text{ and } \right. \right. \\
& \quad \quad \left. \left. \int_0^{\infty} u(t) e^{-t} t^j dt = 0 \text{ for all } j = 0, 1, \dots, k-1, \right\} \right\},
\end{aligned}$$

and the mappings φ_{\pm} again depend on $a \left(\frac{t-i}{t+i} \right)^{-k}$ and $b \left(\frac{t-i}{t+i} \right)^k$.

Proof. It follows immediately from the representations (21)–(23). \blacksquare

Theorem 3.2 can also be used to derive representations for the cokernels of the operators $W(a) \pm H(b)$ in the situation where $(\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{N}$. Indeed, recalling that for $p \in [1, \infty)$, the adjoint operator $(W(a) \pm H(b))^*$ can be represented in the form (4) and (\bar{d}, \bar{c}) is the subordinated pair for (\bar{a}, \bar{b}) , one can observe that the operators $W(\bar{d})$ and $W(\bar{c})$ are also Fredholm and

$$\text{ind } W(\bar{d}) = -\kappa_2, \quad \text{ind } W(\bar{c}) = -\kappa_1,$$

so $(-\kappa_2, -\kappa_1) \in \mathbb{Z}_- \times \mathbb{N}$. Therefore, Theorem 3.2 applies and one can formulate the following result.

Theorem 3.3 *Let $(\kappa_1, \kappa_2) \in \mathbb{Z}_- \times \mathbb{N}$, and let $m \in \mathbb{N}$ satisfy the requirement*

$$1 \geq 2m - \kappa_2 \geq 0.$$

Then

(i) *If κ_2 is odd, then*

$$\begin{aligned} \text{coker } (W(a) \pm H(b)) &= W \left(\left(\frac{t-i}{t+i} \right)^{-m} \right) \\ &\times \left(\left\{ \frac{1 \mp \sigma(\bar{d})}{2} W(\overline{d_{-1}}) \{ \mathbb{C}\psi_0 \} \dot{+} \varphi_{\pm}(\text{im } \mathbf{P}^{\pm}(\bar{c})) \right\} \cap \text{im } W \left(\left(\frac{t-i}{t+i} \right)^m \right) \right). \end{aligned}$$

(ii) *If κ_2 is even, then*

$$\begin{aligned} \text{coker } (W(a) \pm H(b)) &= \\ &= W \left(\left(\frac{t-i}{t+i} \right)^{-m} \right) \left(\left\{ \varphi_{\pm}(\text{im } \mathbf{P}^{\pm}(\bar{c})) \right\} \cap \text{im } W \left(\left(\frac{t-i}{t+i} \right)^m \right) \right), \end{aligned}$$

and the mappings φ_{\pm} depend on $\bar{a} \left(\frac{t-i}{t+i} \right)^{-m}$ and $\tilde{\bar{b}} \left(\frac{t-i}{t+i} \right)^m$.

3.2 The Case II: $\nu(c) \neq 0$ and $\nu(d) \neq 0$.

According to Theorem 2.1, the operators $W(c)$ and $W(d)$ are one-sided invertible. In this situation the pair $(W(c), W(d))$ belongs to one of the classes (r, r) , (l, l) , (l, r) or (r, l) , where letter r or l means that the corresponding operator is right- or left-invertible. It is worth mentioning that if the pair $(W(c), W(d))$ belongs to

the class (r, l) , then the operator $W(a) + H(b)$ is normally solvable but it is not semi-Fredholm. Further, if $(W(c), W(d)) \in (l, r)$ then, generally, it is not known whether $W(a) + H(b)$ is normally solvable or not. If $(W(c), W(d))$ belongs to one of the classes (r, r) or (r, l) , then the kernels of the operators $W(a) + H(b)$ and $W(a) - H(b)$ can be described using results of Section 2. For the description of the cokernels of the operators $W(a) + H(b)$ and $W(a) - H(b)$ in the cases (l, l) and (r, l) , one has to assume that $p \in [1, \infty)$ and use the relation (4). If $(W(c), W(d))$ belongs to the class (r, l) , then one can proceed similarly to Subsection 3.1. More precisely, we have to consider three situations, namely,

- (i) The index $\nu(c) < 0$ and the index $n(c) > 0$.
- (ii) The index $\nu(c) < 0$ and the index $n(c) = 0$.
- (iii) The index $\nu(c) < 0$ and the index $n(c) < 0$.

Since in this situations, the operator $W(c)$ is right-invertible, the kernels of the operators $W(a) + H(b)$ and $W(a) - H(b)$ can be described by Proposition 1.1 and subsequent use of Theorems 2.3, 2.4 and 2.5.

As was already mentioned, if the pair $(W(c), W(d))$ belongs to the class (l, r) , then it is not known whether the operators $W(a) \pm H(b)$ are normally solvable. Nevertheless, the kernels and cokernels of these operators still can be described. However, it is worth noting that Proposition 1.1 cannot be directly used. Thus let us sketch the idea how to proceed in this situation. We have to deal with the following cases

- (i) The index $\nu(c) > 0$ and the index $n(c) > 0$.
- (ii) The index $\nu(c) > 0$ and the index $n(c) = 0$.
- (iii) The index $\nu(c) > 0$ and the index $n(c) < 0$.

In these situations the operators $W(a) \pm H(b)$ admit the factorization

$$\begin{aligned}
 W(a) \pm H(b) &= \left(W \left(ae^{-i\nu t/2} \left(\frac{t-i}{t+i} \right)^{-k} \right) \pm H \left(be^{i\nu t/2} \left(\frac{t-i}{t+i} \right)^k \right) \right) \\
 &\quad \times W \left(e^{i\nu t/2} \left(\frac{t-i}{t+i} \right)^k \right), \\
 W(a) \pm H(b) &= (W(ae^{-i\nu t/2}) \pm H(be^{i\nu t/2})) W(e^{i\nu t/2}), \\
 W(a) \pm H(b) &= (W(ae^{-i\nu t/2}) \pm H(be^{i\nu t/2})) W(e^{i\nu t/2}),
 \end{aligned}$$

where $\nu = \nu(c)$ and k are defined as in Subsection 3.1. Let us consider, for definiteness, the operator $W(a) + H(b)$ and note that the respective subordinated pairs for the first operators in the right-hand sides of the last representations are

$$\left(ce^{-i\nu t} \left(\frac{t-i}{t+i} \right)^{-2k}, d \right), (ce^{-i\nu t}, d), \text{ and } (ce^{-i\nu t}, d)$$

with the respective indices ν and n defined as

$$\begin{aligned} \nu \left(ce^{-i\nu t} \left(\frac{t-i}{t+i} \right)^{-2k} \right) &= 0 \text{ and } n \left(ce^{-i\nu t} \left(\frac{t-i}{t+i} \right)^{-2k} \right) = -2k + n(c), \\ \nu (ce^{-i\nu t}) &= 0 \text{ and } n (ce^{-i\nu t}) = 0, \\ \nu (ce^{-i\nu t}) &= 0 \text{ and } n (ce^{-i\nu t}) = n(c). \end{aligned}$$

Now using the corresponding results of Section 2 and those obtained in Subsection 3.1, one can get a complete description for the kernels and cokernels of the operators $W(a) + H(b)$ and $W(a) - H(b)$.

3.3 The Case III.

Assume that the only one of the indices $\nu(c)$ or $\nu(d)$ is equal to zero. This case can be handled similarly to the Cases I and II without any new features. Therefore, we omit detailed formulations here. However, it is worth mentioning that in this case the operators $W(a) \pm H(b)$ are semi-Fredholm but not Fredholm.

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